

# PHASE TRANSITION IN OPINION DIFFUSION IN SOCIAL NETWORKS

*Lin Li*<sup>†</sup>

*Anna Scaglione*<sup>†</sup>

*Ananthram Swami*<sup>‡</sup>

*Qing Zhao*<sup>†</sup>

<sup>†</sup> ECE Department, University of California, Davis, CA, USA

<sup>‡</sup> Army Research Laboratory, Adelphi, MD, USA

## ABSTRACT

Gossiping models have been increasingly applied to study social network phenomena, in particular, to model the dynamics of social behavior or belief through local interactions. In this context, this paper investigates how the opinions of social agents diffuse in a network under a so-called *hard-interaction* model, in which the agents interact more strongly with neighbors that share their beliefs and have no influence on the neighbors whose opinions differ by more than a threshold. We analyze the convergence properties of the opinion dynamics and provide analytical insights to characterize the phase transition from a society of radicalized opinions to one of convergent behavior.

**Index Terms**— opinion diffusion, opinion dynamics, social networks, phase transition, herding.

## 1. INTRODUCTION

The study of the convergence of social behavior can be found in many fields. Examples include, but are not limited to the *herding behavior* [1] in Economics, the *fad and trend* behaviors [2] in Social Psychology, and the *Bandwagon effect* [3] in political science. A number of different approaches to elucidate these phenomena have emerged. Two prominent classes of models have been studied extensively, namely, *Bayesian* models and *non-Bayesian* models. Bayesian models [1, 2] view individuals as *rational agents*: opinions (or beliefs) of agents are probabilities of a given state, conditioned on all the available information; opinions are updated using Bayes rule, with information communicated through neighbors' actions. Bayesian models focus on the mechanism of propagating information from one agent to another for optimal decision-making. However, because of the complexity required to describe the diffusion of opinions in a network of rational agents only simple sequential interactions can be fully analyzed. Models based on sequential interactions usually assume that agents enter the society/market one by one, and each makes an irreversible decision by only observing the actions of its predecessors. This assumption can be well justified in certain specific cases, but it is hard to generalize to random interactions between agents in a network. This motivates the study of non-Bayesian models [4, 5, 6, 7, 8, 9, 10] which use simple and heuristic local belief updating rules to characterize agents' interactions in a network. Non-Bayesian models aim to capture the opinion dynamics in a network and model how the initial opinions and the underlying social network structure affect the alignment of social behavior. Earlier non-Bayesian formulations include [4, 5] which model the interactions based on simple synchronous *linear* updates: individuals assign suitable weights to their interacting agents on the basis of relative importance. The updating rule has the same form as

the well studied average consensus algorithm [11], and thus the convergence analysis is straight forward. Another class of non-Bayesian model is the so called *Hegselmann-Krause* (HK) model [7, 8, 9], in which agents update their opinions using a *nonlinear* model: interactions are performed through a synchronous update by averaging all the opinions that differ by less than a confidence level  $\epsilon$ ; the rate of change in opinions is determined by a convergence parameter  $\mu$  (that is constant in time and across the network). Similar studies have also explored the effects of simple interactions between two neighboring agents. For example, Deffuant et al. [10] modeled the network on a square grid, in which the agents can only communicate with their four immediate neighbors and exchange their beliefs with a fixed weight parameter if the distance in opinion is smaller than a given threshold. Weisbuch [12] extended this simple lattice topology to a scale free network model and also examined the convergence property using a heterogeneous constant threshold. The focus of these papers is on the modeling of social interactions and on how to reach opinion consensus. Analysis of the above nonlinear models is generally carried out by extensive computer simulations, but explicit mathematical results are limited.

In this paper, we explore a theoretical framework called the *hard-interaction* model, to provide analytical insights on the asymptotic behavior of a social group in relation to the agents' initial opinion profile and the underlying network structure. The *hard-interaction* model generalizes the binary decision making (decision between two alternatives) as assumed in [7, 8, 9, 12] to the multi-alternative decision making (decision between multiple alternatives). Specifically, rather than restricting the opinions to lie in a bounded (real) interval, we treat each agent's opinion as a vector of probabilities; each element of the opinion vector represents the probability that a certain alternative is true. Furthermore, we extend the HK model by introducing a trust function  $\rho$  similar in spirit to the convergence parameter  $\mu$  defined in [7, 8, 9, 10, 12], but allowing the trust function  $\rho$  to vary with the opinion distance between the interacting agents. Hence,  $\rho$  is time varying and its value depends on how the distance is defined. Finally, we let the underlying social network to be any arbitrarily connected network. Agent interactions are pairwise random encounters. Opinion updates are as follow: the distance between the opinions of two interacting agents decreases only if the previous distance between their opinions is smaller than a threshold  $\tau$ , otherwise the opinions remain unchanged; the rate of change of the opinion distance is governed by the time-varying function  $\rho$ . (Note that if  $\rho$  is kept constant and  $L_1$ -norm is used to measure the opinion distance, then our model is analogous to the HK model for binary decision making.) The contribution of this paper is three-fold: (i) we provide analytical insights on the asymptotic behavior of the social group and show the existence of a phase transition from diverging clusters of opinions to herding; (ii) we prove that a necessary condition for society to herd with probability 1 is that the threshold  $\tau$  be strictly greater than the expected initial opin-

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ion distance, averaged over the edge probability distribution; (iii) we also show that if the social fabric, represented by the probability distribution of pairwise communications, is random and independent of the initial belief distribution, then the social fabric does not effect the phase transition towards herding. All our findings are validated numerically.

## 2. INTERACTION MODEL

In our model there are  $\mathcal{V} = \{1, 2, \dots, n\}$  social agents. They can interact with other agents at random, but only if they are connected through an edge of a fixed communication graph  $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}_c)$ , where  $\mathcal{E}_c$  is the set of edges. Throughout the paper,  $\mathcal{G}_c$  is assumed to be connected and  $\mathcal{N}_j$  denotes the set of neighbors of agent  $j$ , whose cardinality  $|\mathcal{N}_j|$  is called the degree of agent  $j$ . We model the  $j$ th agent's opinion as a  $q$ -dimensional vector  $\mathbf{x}_j = [x_{j1}, \dots, x_{jq}]$  in  $\mathcal{X} = \{\mathbf{x} = [x_1, \dots, x_q] \mid \sum_{\ell=1}^q x_{\ell} = 1 \text{ and } x_{\ell} \in [0, 1]\}$ . Although this is not critical for the technical derivations, we can interpret  $x_k$  as a conditional probability  $P(\theta_k | \xi_i)$ , where  $\theta_1, \dots, \theta_q$  are  $q < n$  possible outcomes of an experiment  $\Theta$ , and  $\xi_i$  is the  $i$ th agent's private information on  $\Theta$ , generated from a certain probability measure  $P(\xi | \theta)$ . The vector  $\mathbf{x}_j[0]$  is the belief of agent  $j$  prior to any interaction, while  $\mathbf{x}_j[k]$  represents the belief after  $k$  interactions. In our simulations, the initial beliefs  $\mathbf{x}_j[0]$  of the agents are drawn from a uniform distribution over  $\mathcal{X}$ .

We introduce a proper distance function  $d(\mathbf{x}_i, \mathbf{x}_j) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+$  ( $\mathbb{R}^+$  is the set of real non-negative numbers) to measure the degree of agreement between agents  $i$  and  $j$ . With respect to the norm  $\|\mathbf{x}_j\| := d(\mathbf{x}_j, \mathbf{x}_j)$ , we assume that the set  $\mathcal{X}$  is bounded, i.e.,  $\sup_j \|\mathbf{x}_j\| < \infty \forall \mathbf{x}_j \in \mathcal{X}$  and  $\forall j \in \mathcal{V}$ . Hence,  $d(\mathbf{x}_i, \mathbf{x}_j) \leq 2\sup_i \|\mathbf{x}_i\| := d_{\max}$  using the triangular inequality. Agents in the network interact at random and change their beliefs. We define a time-invariant vector  $\mathbf{p}$  whose  $i$ th element  $p_i$  is the probability of node  $i$  initiating an interaction and the stochastic matrix  $\mathbf{P}$  whose  $(i, j)$ th element  $P_{ij}$  denotes the probability that node  $i$  will choose to interact with node  $j$ . Matrix  $\mathbf{P}$  is assumed to have the same structure as  $\mathcal{G}_c$ : if an edge  $(i, j)$  does not exist in  $\mathcal{G}_c$ , then  $P_{ij} = 0$ . We assume  $P_{ii} = 0$  and that the matrix  $\mathbf{P}$  is fixed. Let  $\mathbf{x}'$  and  $\mathbf{x}''$  denote the variables after a generic update and  $d_{ij}[k]$  denotes the distance  $d(\mathbf{x}_i[k], \mathbf{x}_j[k])$  after  $k$  network-wise interactions have occurred. It is assumed that *the distance between beliefs after an update cannot be larger than that before the update*. Formally, the degree of change in belief is captured by a discontinuous function  $\rho(d)$  according to the following nonlinear model:

$$(a1) \quad d_{ij}[k+1] = (1 - \epsilon_k \rho(d_{ij}[k])) d_{ij}[k], \quad (1)$$

where  $\epsilon_k$  denotes the step-size. For technical reasons, we need

$$(a2) \quad \epsilon_k : \sum_{k=1}^{\infty} \epsilon_k \rightarrow \infty, \quad \sum_{k=1}^{\infty} \epsilon_k^2 \leq \infty.$$

One example of an interaction that leads to (1) is  $d(\mathbf{x}'_i, \mathbf{x}'_j) = d(\mathbf{x}_i, \mathbf{x}_j) - d(\mathbf{x}_i, \mathbf{x}'_i) - d(\mathbf{x}_j, \mathbf{x}'_j)$ , which geometrically means that the beliefs are moving closer through the shortest path connecting them. Regarding (1), the following assumptions hold.

(a3)  $\rho(d)$  is a non-increasing function of  $d$ .

(a4)  $\rho(d)$  is  $C^2$ -differentiable and  $0 < \rho(d) \leq 1/\epsilon_k \forall d \in [0, \tau]$ .

(a5)  $\tau : \forall d \geq \tau \rightarrow \rho(d) = 0$ .

(a6)  $\rho(0)/\rho(\tau^-) \leq \beta < \infty$ .

(a7)  $h(d) = \rho(d)d$  is concave  $\forall d \in [0, \tau]$ .

Specifically,  $\rho(d)$  represents the degree of change in opinion distance after each interaction. (a3) implies that agents interact strongly (i.e., large displacement of belief) with neighbors that share their beliefs, and (a5) indicates that agents have no influence on the neighbors whose opinions differ by more than  $\tau$ , where  $\tau$  is

a measure of how openminded a society is. For the communication model, we have  $P_{ij} > 0 \leftrightarrow (i, j) \in \mathcal{E}_c$ . Define  $\bar{\mathbf{P}}$  to be an  $n \times n$  matrix containing the probability that the pair  $(i, j)$  performs an exchange, i.e.,  $\bar{P}_{ij} = p_i P_{ij} + p_j P_{ji}$ . The *uniform* communication model corresponds to the selection of homogenous rates  $p_i = 1/n$ , and  $P_{ij} = 1/|\mathcal{N}_i|$ , uniform across neighbors  $\forall k$ .

We say that the network attains *consensus (herding)* iff  $\forall i, j \in \mathcal{V}, d(\mathbf{x}_i[k], \mathbf{x}_j[k]) = 0$  for some  $k$ . This does not imply, however, that all agents are certain about a specific outcome of an experiment  $\Theta$ , i.e.,  $\lim_{k \rightarrow \infty} x_{i,\ell}[k] = x_{\ell}^{\infty} \neq \delta(\ell - \ell^*)$  for some  $\ell^* \in [1, \dots, q]$ . In other words, consensus is achieved when all agents have the same belief vector w.r.t the distance measure, but need not believe in only one outcome. If network-wide consensus is not reached, it is possible that the network evolves into non-interacting sub-networks (herds), each of which is internally in consensus; we call such a process *radicalization*.

## 3. ANALYSIS

Under (a2), using stochastic approximation theory [13], we can map (1) onto the *ordinary differential equation* (ODE),

$$\dot{d}_{ij} = -\rho(d_{ij})d_{ij}, \quad (2)$$

in which  $\dot{d}_{ij}$  is the derivative of  $d_{ij}$  with respect to a continuous variable  $t$  replacing the discrete index  $k$ . For the sake of notational convenience, we do not explicitly show that time variable  $t$  in (2) and the rest of this section. Let  $\bar{d}$  be the average of  $d_{ij}$  over the edges  $(i, j) \in \mathcal{E}_c$ , i.e.,  $\bar{d} := \sum_{(i,j) \in \mathcal{E}_c} \bar{P}_{ij} d_{ij}$ . Then, using (2), we get

$$\dot{\bar{d}} = -\sum_{(i,j) \in \mathcal{E}_c} \bar{P}_{ij} \rho(d_{ij}) d_{ij}. \quad (3)$$

Let  $S = \{(i, j) \in \mathcal{E}_c \mid d_{ij} \leq \bar{d}\}$ . Under (a3) and  $\forall (i, j) \in S, \rho(d_{ij}) \geq \rho(\bar{d})$  and thus  $\dot{\bar{d}} \leq -\left[\sum_{(i,j) \in S} \bar{P}_{ij} d_{ij}\right] \rho(\bar{d})$  which equals to zero when either  $\rho(\bar{d}) = 0$  or  $\sum_{(i,j) \in S} \bar{P}_{ij} d_{ij} = 0$ . If both are positive, then the system has to converge. However, as  $S$  changes dynamically, deriving a sufficient condition is not trivial: in remark 1, we show through a counterexample that  $\rho(\bar{d}) > 0$  is not a sufficient condition for convergence. We next establish a lower bound on the rate change  $\dot{\bar{d}}$ .

**Property 1** Under (a3), (a5) – (a7), when  $\bar{d} < \tau$ , the system in (3) is lower bounded by  $\dot{\bar{d}} \geq -\beta \rho(\bar{d}) \bar{d}$ .

**Proof** See Appendix. ■

Based on the above result, it is expected that (3) will not converge if the lower bound does not converge. For convenience, we use  $b$  instead of  $\bar{d}$  to denote the distance for the lower bound system whose dynamic is expressed as  $\dot{b} = -\beta \rho(b)b$ . In fact, the dynamic of  $b$  locally resembles the form of the logistic equation which is investigated in the next lemma.

**Lemma 1** Assume that the network is connected. Under (a1) – (a7), the system  $\dot{b} = -\beta \rho(b)b$  converges if  $\tau > b(0)$ .

**Proof** See Appendix. ■

Given that (3) is lower bounded by  $-\beta \rho(\bar{d}) \bar{d}$  and  $\mathcal{G}_c$  is connected, we use the previous lemma to establish a necessary condition for (3) to converge under the proposed model in (1).

**Lemma 2** Assume that the network is connected. Under (a1) – (a7), a necessary condition for the system in (3) to converge almost surely is  $\tau > \bar{d}[0]$ .

**Proof** From Lemma 1, the system  $\dot{b} = -\beta\rho(b)b$  converges if  $\tau > b(0)$ . Because (3) is lower bounded by this system, given a threshold  $\tau$ , (3) will not converge if  $\tau \leq \bar{d}(0)$ , or  $\bar{d}[0]$  w.r.t. the discrete-time system in (1). ■

Lemma 2 indicates that the system will converge if the threshold is above  $\bar{d}[0]$ , the average initial distance between agent pairs that can interact. In other words, if the system is open-minded enough (relative to this initial dissonance in opinions), then the system will converge. Interestingly, what we observe from the numerical results (shown in the next section) is that the interaction model exhibits a phase transition from radicalization to herding whenever  $\tau$  is chosen sufficiently above  $\bar{d}[0]$ . The following remark is in order:

**Remark 1** Let there be two groups  $H_1$  and  $H_2$ , where agents have opinions with zero distance within each group, but  $\forall(i, j) : i \in H_1, j \in H_2, d_{ij} = \tau + \epsilon$ . Then if  $\sum_{(i,j) \in H_1 \times H_2} \bar{P}_{ij} < \frac{\tau}{\tau + \epsilon}$ , it is easy to see that  $\bar{d}[0] < \tau$ . This network cannot converge because the two groups do not communicate, as the rate  $\rho(d_{ij}) = 0 \forall(i, j) : i \in H_1, j \in H_2$ . However,  $\rho(\bar{d}[0]) > 0$  which proves that it is not a sufficient condition for herding. Note that in this case,  $\sum_{(i,j) \in S} \bar{P}_{ij} d_{ij}[0] = 0$ .

#### 4. HOW COMMUNICATION RATES AFFECT HERDING

In the following lemma, we show that social fabrics will, on average, exhibit the same phase transition.

**Lemma 3** Let  $d_c[0]$  be the expected initial opinion distance between any two agents in the set  $\mathcal{V}$ . If the connected graph  $G_c$  and  $\bar{P}_{ij}$  are random, and they are independent of the initial opinion distance  $d_{ij}[0]$ , then  $\mathbb{E}[\bar{d}[0]] = d_c[0]$ , with respect to  $G_c$  and the initial distance distribution.

**Proof** See Appendix. ■

In our simulations, we will verify this analytical result and compare it with a heuristic that seeks to decrease the initial opinion distance  $\bar{d}[0]$ . Specifically, we re-distribute the communication rates by uniformly assigning positive interaction rate to the neighboring agents  $(i, j) \in \mathcal{E}_c$  who have similar initial beliefs (below the threshold) and zero interaction rate to the agents whose initial beliefs are above the threshold. Formally, denote by  $\mathcal{M}_i$  the neighbors of agent  $i$  in the set  $\mathcal{E}_{\text{eff}}[0] = \{(i, j) \in \mathcal{E}_c | d_{ij}[0] < \tau\}$  and  $|\mathcal{M}_i|$  its cardinality. The uniform communication over  $\mathcal{E}_{\text{eff}}[0]$  is defined as

$$p_i = \frac{1}{n}, \quad P_{ij} = \begin{cases} \frac{1}{|\mathcal{M}_i|} & \text{if } (i, j) \in \mathcal{E}_{\text{eff}}[0], \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

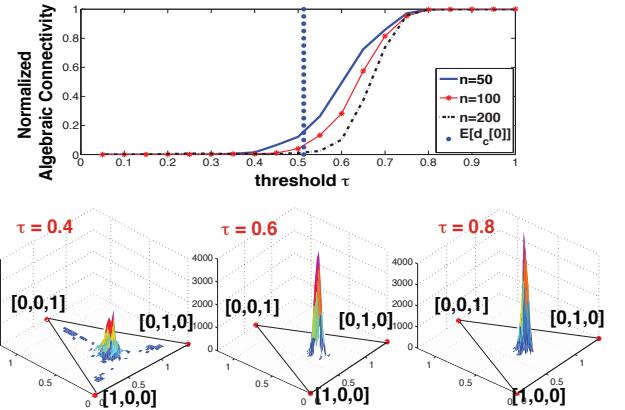
This redistribution causes the interaction probability to be correlated with the opinion distance resulting in a twofold benefit: it decreases the expected initial distance  $\bar{d}[0]$  which is closely related to the critical threshold, and the communications are not wasted on the agents that have little influence on each other, if at all.

#### 5. SIMULATIONS

##### 5.1. Phase Transition

Fig. 1 shows the asymptotic algebraic connectivity of a graph  $\mathcal{G}_{\text{eff}}[k] := \{\mathcal{V}, \mathcal{E}_{\text{eff}}[k]\}$  where  $\mathcal{E}_{\text{eff}}[k] = \{(i, j) \in \mathcal{E}_c | d_{ij}[k] < \tau\}$ , w.r.t.  $\tau$  for  $n = 50$ ,  $n = 100$  and  $n = 200$  after  $k = 2000$  network-wise interactions. Distance is measured in the  $L_2$ -norm

and  $\mathcal{G}_c$  is fully connected. (Note that Lemma 1 and 2 hold for any connected network. The choice of using a fully connected network is arbitrary.) The plot is generated by averaging over 300 runs. Each starts with an uniformly distributed random initial beliefs and ends after 2000 random interactions using the uniform communication rate (i.e.,  $p_i = 1/n$  and  $P_{ij} = 1/(n-1)$ ) and a function  $\rho(\bar{d})$  which equals to 1 for  $0 \leq \bar{d} < \tau$  and zero otherwise. Note that  $\beta = 1$  in this case. From Fig. 1, one observes that there exists a phase transition from a society of radicalized opinions to a society with consistent opinion at a critical threshold. The network converges with probability one if  $\tau$  is above a threshold which is observed to be around 0.75. We have learned, from Lemma 2, that a necessary condition for the system to converge is that  $\tau > \bar{d}[0]$ , where  $\bar{d}[0]$  is the expected opinion distance at  $k = 0$  and is represented by the dotted line. For all  $\tau < \bar{d}[0]$ , with a high probability, the system will not converge as  $n$  increases.



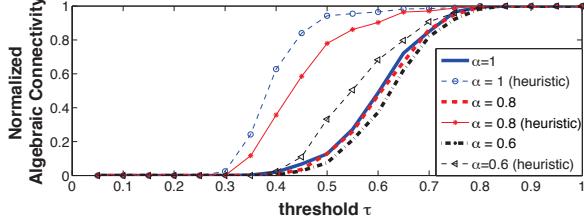
**Fig. 1.** Phase transition (top) of a society from radicalized beliefs to a consistent belief. Histogram (bottom) of 300 asymptotic opinion profiles with  $n = 100$  at  $\tau = 0.4, 0.6, 0.8$ .

##### 5.2. Rate of Interaction

Fig. 2 compares the phase transitions for the original uniform rate of interaction in  $\mathcal{E}_c$  and the heuristic described in (4) over three random geometric networks (RGG) of  $n = 50$  agents. (Note that the choice of RGG is arbitrary.) Each has a distinct radius of communication  $\alpha \in [0, 1]$ . In particular,  $\alpha = 1$  represents a fully connected  $\mathcal{G}_c$  and  $\alpha = 0$  corresponds to a fully disconnected network. Distance is measured in the  $L_2$ -norm. The plot is generated by averaging over 600 runs with uniformly distributed random initial beliefs over  $\mathcal{X}$  of dimension 3 after  $k = 2000$  random interactions. Observe that the performances for  $\alpha = 1, 0.8$  and  $0.6$  are similar when the underlying network is connected and independent of  $d_{ij}[0]$ . It is consistent with our findings, given in Lemma 2 and Lemma 3. In contrast, when  $\bar{P}_{ij}$  is correlated with  $d_{ij}[0]$  through the rate re-distribution (4), the probability of forming a convergent belief increases. For example, with  $\tau = 0.5$  in a fully connected network, i.e.,  $\alpha = 1$ , the probability of the agents forming a convergent belief increases from 0.15 to 0.95.

#### 6. CONCLUSION

We proposed a *hard-interaction* model for social networks and analyzed its convergence properties in terms of the expected initial opinion distance. By describing the opinion dynamics as the change in distance between opinions, we demonstrated the existence of a phase

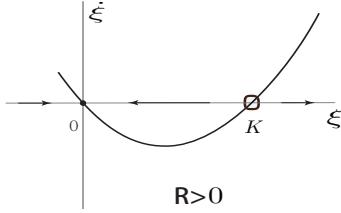


**Fig. 2.** Performance comparison between uniform random and the heuristic communication-rate schemes.

transition w.r.t. the opinion threshold and proved that the critical threshold is lower bounded by the expected initial opinion distance.

## 7. APPENDIX

**Property 1:** Let  $\mathcal{E}_{\text{eff}} = \{(i, j) \in \mathcal{E}_c \mid d_{ij} < \tau\}$  and  $\mathcal{E}_{\text{eff}}^c = \{(i, j) \in \mathcal{E}_c \mid d_{ij} \geq \tau\}$  be the complement of  $\mathcal{E}_{\text{eff}}$  in  $\mathcal{E}_c$ . Define  $P_{\text{eff}} := \sum_{(i, j) \in \mathcal{E}_{\text{eff}}} \bar{P}_{ij}$  and  $\bar{d}_{\text{eff}} := \sum_{(i, j) \in \mathcal{E}_{\text{eff}}} \bar{P}_{ij} d_{ij}$ . Under (a7), Jensen's inequality and the relation  $\bar{d}_{\text{eff}} \leq \bar{d}$  yield  $\sum_{(i, j) \in \mathcal{E}_{\text{eff}}} \frac{\bar{P}_{ij}}{P_{\text{eff}}} h(d_{ij}) \leq \frac{\bar{d}_{\text{eff}}}{P_{\text{eff}}} \rho\left(\frac{\bar{d}_{\text{eff}}}{P_{\text{eff}}}\right) \leq \frac{\bar{d}}{P_{\text{eff}}} \rho(\bar{d}_{\text{eff}})$ . Since  $\rho(d_{\ell m}) = 0$  for  $\forall (\ell, m) \in \mathcal{E}_{\text{eff}}^c$ , hence  $\sum_{(i, j) \in \mathcal{E}_c} \bar{P}_{ij} h(d_{ij}) = \sum_{(i, j) \in \mathcal{E}_{\text{eff}}} \bar{P}_{ij} h(d_{ij}) \leq \bar{d} \rho(\bar{d}_{\text{eff}})$ . Since  $\bar{d}_{\text{eff}} < \bar{d} < \tau$ , under (a3) and (a6),  $\rho(\bar{d}_{\text{eff}})/\rho(\bar{d}) \leq \rho(0)/\rho(\tau^-) \leq \beta$ . Hence,  $\sum_{(i, j) \in \mathcal{E}_c} \bar{P}_{ij} h(d_{ij}) \leq \beta \rho(\bar{d}) \bar{d}$ .



**Fig. 3.** Vector Field: the point  $\xi(s; t)$  moves toward the stable fixed point  $\xi_1 = 0$  if  $\xi(0; t) < K$ . Otherwise, it moves away from  $\xi_1$ .

**Lemma 1:** Suppose that  $b(t+s)$  is in a neighborhood of  $b(t)$  provided that  $s$  is small. Hence, Taylor's expansion gives  $\rho(b(t+s)) \approx \rho(b(t)) + (b(t+s) - b(t)) \dot{\rho}(b(t))$ . Then system  $b(t+s) = -\beta \rho(b(t+s)) b(t+s)$  becomes

$$\dot{b}(t+s) = \begin{cases} -\beta \rho(b(t)) b(t+s) & \text{if } \dot{\rho}(b(t)) = 0; \\ -f(s; t) & \text{if } \dot{\rho}(b(t)) < 0. \end{cases}$$

where  $f(s; t) = \beta \left[ \rho(b(t)) - b(t) \dot{\rho}(b(t)) \left(1 - \frac{b(t+s)}{b(t)}\right) \right] b(t+s)$  and  $\dot{\rho}(b) := d\rho/db$ . In the first case,  $\rho(b(t))$  is locally constant and hence, the local rate of convergence around  $b(t)$  is exponential and is equal to  $\rho(b(t))$  when  $\rho(b(t)) > 0$ , i.e.,  $b(t) < \tau$ . For the second case when  $\dot{\rho}(b(t)) < 0$ , define  $\xi(s; t) := b(t+s)/b(t)$ . With respect to  $s$ , the dynamics of  $\xi$  become

$$\dot{\xi}(s; t) = \frac{1}{b(t)} \dot{b}(t+s) = -\beta R(t) \xi(s; t) \left(1 - \frac{\xi(s; t)}{K(t)}\right), \quad (5)$$

$$\text{where } R(t) = \rho(b(t)) - b(t) \dot{\rho}(b(t)), \quad (6)$$

$$K(t) = 1 + \frac{\rho(b(t))}{-b(t) \dot{\rho}(b(t))} > 1. \quad (7)$$

Note that the dynamics of  $\xi(s; t)$  resemble the logistic equation. There exists two equilibria at  $\xi_1 = 0$  (stable) and  $\xi_2 = K(t)$  (unstable). When  $R(t) > 0$ , as shown in Fig. 3, the system will converge if  $K(t) > \xi(0; t) = 1$ . Indeed, if  $b(t) < \tau$ , then  $K(t) =$

$1 - \frac{\rho(b(t))}{b(t) \dot{\rho}(b(t))} > 1$ . and (5) converges. The exponential rate of convergence equals  $R(t)$ . Moreover, since  $b(t)$  is a monotonically decreasing function of  $t$ , an equivalent condition for convergence is  $b(0) < \tau$ . On the contrary, when  $b(t) = \tau$ , we get  $K(t) = 1$  because  $\rho(\tau) = 0$ . For  $b(t) > \tau$ , both  $\rho(b(t))$  and  $\dot{\rho}(b(t))$  are zero, which implies  $R(t) = 0$  in (5) and bifurcation occurs. Thus the system may not converge when  $b(t) \geq \tau$  for  $\forall t$ .

**Lemma 3:** Define  $d_c[0] := \mathbb{E}\{d_{ij}[0]\}$  to be the expected initial distance between any agents in  $\mathcal{V}$ . Since the rates  $\bar{P}_{ij}$  are random and independent of  $d_{ij}[0]$ , then

$$\begin{aligned} \mathbb{E}[\bar{d}(0)] &= \mathbb{E}\left[\sum_{(i,j) \in E_c} \bar{P}_{ij} \mathbb{E}[d_{ij}[0] | \bar{P}_{ij}]\right] \\ &= \mathbb{E}\left[\sum_{(i,j) \in E_c} \bar{P}_{ij}\right] d_c[0] = d_c[0]. \end{aligned}$$

## 8. REFERENCES

- [1] C.P. Chamley, *Rational Herds: Economic Models of Social Learning*, Cambridge University Press, 2004.
- [2] S. Bikhchandani, D. Hirshleifer, and I. Welch, "A theory of fads, fashion, custom, and cultural change as informational cascades," in *J. Pol. Econ.*, 1992, vol. 100, pp. 992–1026.
- [3] H. Leibenstein, "Bandwagon, snob, and veblen effects in the theory of consumers' demand," in *Quarterly Journal of Economics*, 1950, vol. 64, pp. 183–207.
- [4] Jr.P. French, "A formal theory of social power," in *Psychological Review*, 1956, vol. 63, pp. 181–194.
- [5] M.H. DeGroot, "Reaching a consensus," in *Journal of American Statistical Association*, 1974, vol. 69, pp. 118–121.
- [6] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, "Mixing beliefs among interacting agents," in *Adv. Compl. Syst.*, 2000, vol. 3, pp. 87–98.
- [7] G. Weisbuch, G. Deffuant, F. Amblard, and J.P. Nadal, "Interacting agents and continuous opinions dynamics," in *Heterogeneous agents, interactions, and economic performance*, 2001.
- [8] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis and simulations," in *Journal of Artificial Societies and Social Simulation*, 2002, vol. 5.
- [9] V.D. Blondel, J.M. Hendrickx, and J.N. Tsitsiklis, "On Krause's multi-agent consensus model with state-dependent connectivity," in *IEEE Trans. Auto. Control*, 2009, vol. 54, pp. 2586–2597.
- [10] G. Deffuant, F. Amblard, and G. Weisbuch, "How can extremism prevail? a study based on the relative agreement interaction model," *Journal of Artificial Societies and Social Simulation*, vol. 5, 2002.
- [11] J.N. Tsitsiklis, *Problems in decentralized decision making and computation*, Ph.D. thesis, Dept. of Electrical Engineering and Computer Science, M.I.T., Boston, MA, 1984.
- [12] G. Weisbuch, "Bounded confidence and social networks," *The European Physical Journal B - Condensed Matter and Complex Systems*, vol. 38, no. 2, pp. 339–343, 2004-03-25.
- [13] H. Robbins and S. Monroe, "A stochastic approximation method," in *The Annals of Mathematical Statistics*, Sep. 1951, vol. 22, pp. 400–407.